Black hole hydrodynamics

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Abstract

The curved geometry of a spacetime manifold arises as a solution of Einstein's gravitational field equation. We show that the metric of a spherically symmetric gravitational field configuration can be viewed as an *optical metric* created by the moving material fluid with nontrivial dielectric and magnetic properties. Such a "hydrodynamical" approach provides a simple physical interpretation of a horizon.

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INTRODUCTION

Recently, there has been a growing interest in the black hole type geometries which do not arise due to the genuine gravitational interaction but are formed due to the nontrivial (electric and magnetic) properties and dynamics of the matter [1, 2, 3, 4]. When sound or light propagates in such a matter, it feels the effective optical metric and moves along the corresponding geodesics. In particular, the formation of the horizon type structures have been demonstrated in many cases. In other words, the "effective black holes" were discovered in these non-gravitational physical models.

In this paper we look in the opposite direction. Namely, we consider the *genuine black holes* of the gravitational theory and demonstrate that the corresponding metric can be interpreted as the *optical metric* arising due to the inhomogeneous motion of the material fluid in the flat Minkowski spacetime. Such an view can be useful in understanding the characteristic properties of black holes.

In plain words, whereas in [1, 2, 3, 4] the black hole is "fictious" (or effective) arising from the real dynamics of matter, here we consider the real black hole which corresponds to a "fictious" (or effective) dynamics of matter. Since the latter is a fluid, we can call such a model "a hydrodynamics of a black hole".

OPTICAL GEOMETRY

Let us consider a fluid with nontrivial permittivity ε and permeability μ constants which moves in a spacetime with the metric $\widetilde{g}_{ij}$. We will be mainly interested in the case of the Minkowski flat space metric, $\widetilde{g}_{ij} = \eta_{ij}$, however for the sake of generality we will keep $\widetilde{g}_{ij}$ arbitrary. The motion is described by the 4-velocity vector field u^i which is normalized as $\widetilde{u}^2 := u^i u^j \widetilde{g}_{ij} = c^2$.

In such a medium, the light is propagating along the null geodesics not of the original spacetime metric $\eta_{\mu\nu}$ but of the so called optical metric, see [5, 6],

$$g_{ij} = \widetilde{g}_{ij} + (1/n^2 - 1)\, u_i u_j / c^2, \tag{1}$$

where $n = \sqrt{\varepsilon\mu}$ is the refraction index of the fluid. The inverse metric reads

$$g^{ij} = \widetilde{g}^{ij} + (n^2 - 1)\, u^i u^j / c^2. \tag{2}$$

It is worthwhile to note that the velocity has a different normalization for the optical metric: $u^2 := u^i u^j g_{ij} = c^2/n^2$ which is evident from (1).

The kinematic properties of the fluid flow with respect to the metric g_{ij} are described by the tensors of shear σ_{ij}, vorticity ω_{ij}, the volume expansion θ, and the acceleration a^i. These objects are constructed from the components of the covariant derivative $\nabla_k u^i$ of the 4-velocity with the help of the projector $h^i_j = \delta^i_j - u^i g_{jk} u^k/u^2$ as follows: $\sigma_{ij} := h^k{}_i h^l{}_j \nabla_{(k} u^m \, g_{l)m} - \frac{1}{3}\, g_{ik}\, h^k_j \nabla_l u^l$, $\omega_{ij} := h^k{}_i h^l{}_j \nabla_{[k} u^m \, g_{l]m}$, $\theta := \nabla_l u^l$, $a^i := u^k \nabla_k u^i$. If we replace in these formulas $g \rightarrow \widetilde{g}$ (including the replacement of the covariant derivatives $\nabla \rightarrow \widetilde{\nabla}$), we find the kinematics of the fluid with respect to the background spacetime metric. The latter quantities will be denoted by the tildes. It is straightforward to find the relation between the two sets of the kinematic objects, $\omega, \sigma, \theta, a$, and $\widetilde{\omega}, \widetilde{\sigma}, \widetilde{\theta}, \widetilde{a}$.

At first, we notice that the Riemannian connection (Christoffel symbols) for the optical metric (1)-(2) reads:

$$\Gamma^i_{jk} \; = \; \widetilde{\Gamma}^i_{jk} + (1 - n^2)\Big(a^i\, u_j u_k + \frac{1}{3}\,\theta\,(\widetilde{g}_{jk} - u_j u_k)\, u^i \\ + u^i\, \sigma_{jk} + u_j\, \omega_k{}^i + u_k\, \omega_j{}^i\Big). \qquad (3)$$

Note that the indices on the right-hand side are lowered and raised with the help of the background spacetime metric $\widetilde{g}$. Now it is easy to verify that

$$\widetilde{\sigma}_{ij} = \sigma_{ij}, \quad \widetilde{\theta} = \theta, \quad \widetilde{\omega}_{ij} = n^2\, \omega_{ij}, \quad \widetilde{a}^i = n^2\, a^i. \qquad (4)$$

In other words, the kinematic properties of the fluid are basically the same as seen with respect to the spacetime metric $\widetilde{g}$ or with respect to the optical metric g. When the fluid is non-refractive, i.e. $n = 1$, the two metrics, as well as the connection (3) and the kinematic objects, coincide.

SPHERICALLY SYMMETRIC CONFIGURATIONS

Let us consider the Minkowski spacetime with the metric $d\widetilde{s}^2 = \widetilde{g}_{ij}\, dx^i dx^j = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2$ in the spherical coordinate system (t, r, θ, φ). Let the radial flow of the fluid be described by the 4-velocity $u^i = \gamma\,(1, c\beta, 0, 0)$ with

$$\beta = \pm\sqrt{\frac{1 - f(r)}{n^2 - f(r)}}\,, \qquad \gamma = \sqrt{\frac{n^2 - f(r)}{n^2 - 1}}\,. \qquad (5)$$

Then (2) yields the components of the optical metric

$$g^{00} = \frac{1}{c^2}(n^2+1-f), \tag{6}$$

$$g^{01} = \frac{\beta}{c}(n^2-f), \tag{7}$$

$$g^{11} = -f, \tag{8}$$

$$g^{22} = \sin^2\theta\, g^{33} = -\frac{1}{r^2}. \tag{9}$$

The corresponding optical line element reads

$$\begin{aligned} ds^2 &= g_{ij}\,dx^i dx^j = f\,\frac{c^2}{n^2}\,dt^2 + 2\,\frac{c\beta}{n^2}\,(n^2-f)\,dtdr \\ &+\frac{(f-n^2-1)}{n^2}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\varphi^2. \end{aligned} \tag{10}$$

For an arbitrary refraction index n and for any function $f = f(r)$ this interval describes the spherically symmetric static geometry which can be recasted into the simple Schwarzschild form

$$ds^2 = f\,c^2 dt'^2 - \frac{1}{f}\,dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\,d\varphi^2, \tag{11}$$

if we change the original time coordinate to the new one:

$$t' = \frac{1}{n}\left(t \mp \int dr\,\frac{\sqrt{(1-f)(n^2-f)}}{cf}\right). \tag{12}$$

Let us study the motion of light in the optical geometry (10). The light propagates along the null geodesics with the tangent vectors $k^i = \dot{x}^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi})$ in such a way that $g_{ij}k^i k^j = 0$. The most interesting case is represented by the purely radial propagation of a photon which corresponds to $k^2 = k^3 = 0$. As a result, the null condition $g_{00}(k^0)^2 + 2g_{01}k^0k^1 + g_{11}(k^1)^2 = 0$ yields the equation for the radial path:

$$\dot{t} = \frac{-\beta(n^2-f) \pm n}{cf}\,\dot{r}. \tag{13}$$

The plus (minus) sign corresponds to the ray propagating from (to) the origin. Now let us take into account that the function f is directly related to the velocity of the fluid, see (5). From the latter equation we find $f = (1-n^2\beta^2)/(1-\beta^2)$. Inserting this into (13), we obtain that the radial coordinate of the photon changes in time as

$$\frac{dr}{cdt} = \frac{n\beta \pm 1}{n \pm \beta}. \tag{14}$$

When the fluid flows *outwards*, i.e., $\beta > 0$, the *incoming* light (with $\dot{r} < 0$) evidently "freezes", with the vanishing of dr/dt, at the radius r_h defined $n\beta|_{r_h} = 1$. The same takes place for an *outgoing* light in the *inwardly* directed flow, when the light becomes "freezed" at $n\beta|_{r_h} = -1$. Both cases are covered by the condition

$$n^2\beta^2\Big|_{r_h} = 1. \tag{15}$$

In all cases, the "freezing" is absent for the light propagating in the *same* direction as the moving dielectric medium.

There is a clear physical interpretation of the above result [7]. Recall that $\beta = v/c$ for the 3-velocity v of the matter flow, and $v_c = c/n$ is the velocity of light in a medium with refraction index n in the absence of external fields. Using this, we can recast the equation (14) into the form of the relativistic transformation of the velocity:

$$\frac{dr}{dt} = \frac{v \pm v_c}{1 \pm v v_c/c^2}. \tag{16}$$

In other words, the propagation rate of light with respect to the flat Minkowski spacetime (the laboratory reference system) is determined as a relativistic sum of the velocity of light in the medium v_c and the relative velocity v of the fluid (the moving reference frame) The "freezing" of light takes place at a surface on which the velocity of matter v becomes equal to the velocity of light v_c in the medium: the light cannot "overcome" the flow after $r = r_h$.

The above observation thus gives a transparent physical interpretation of the horizon. Below we give two illustrative examples.

Schwarzschild black hole

As a first example we take the Schwarzschild black hole. It is described by

$$f(r) = 1 - \frac{r_0}{r}, \qquad r_0 = 2Gm/c^2. \tag{17}$$

The coordinate transformation (12) then reads explicitly

$$t' = \frac{1}{n}\Bigg[t \mp \frac{r_0}{c}\Bigg(2\sqrt{(n^2-1)r/r_0 + 1} + n\,\ln\frac{\sqrt{(n^2-1)r/r_0+1} - n}{\sqrt{(n^2-1)r/r_0+1} + n}\Bigg)\Bigg]. \tag{18}$$

Note that (18) does not cover the case $n = 1$, when one finds $t' = t \mp \frac{r_0}{c} \log(r/r_0 - 1)$ instead.

As it is clear from (5), the condition (15) of “freezing” is fulfilled when $f = 0$. Geometrically, this corresponds to the Schwarzschild horizon, $r_h = r_0$, see (17).

De Sitter geometry

As a second example, we take the de Sitter spacetime. As it is well known, the de Sitter world has many faces in different coordinate systems. Its static spherically symmetric realization arises for

$$f = 1 - \frac{\Lambda}{3} r^2. \tag{19}$$

Here Λ is the cosmological constant which determines the de Sitter radius. Actually, this is only a particular case of the more general metric with $f = 1 - r_0/r - \Lambda r^2/3$ (Schwarzschild-de Sitter or Kottler metric), but we on purpose limit our attention to the spacetime of constant curvature determined by (19).

The corresponding coordinate transformation (12) is then given explicitly by

$$\begin{aligned} t' \;=\; & \frac{1}{n}\Bigg[t \pm \frac{1}{2c}\sqrt{\frac{3}{\Lambda}}\Bigg(2\sqrt{n^2 - 1 + \Lambda r^2/3} \\ & + n\,\ln\frac{\sqrt{n^2 - 1 + \Lambda r^2/3} - n}{\sqrt{n^2 - 1 + \Lambda r^2/3} + n}\Bigg)\Bigg]. \end{aligned} \tag{20}$$

In this case, the cosmological horizon arises at $r_h = \sqrt{3/\Lambda}$.

DISCUSSION

Hydrodynamical interpretation of the spherically symmetric gravitational field appears to be useful for understanding the physical properties of such configurations. In particular, both the black hole horizons and the cosmological horizons are obtained as the surfaces on which the flow of the (fictious) fluid overpowers the propagation of light. In this sense, gravity becomes encoded into the dynamics of the matter in the flat Minkowski spacetime.

The result obtained confirms the conclusions of the physical models considered in [1, 2, 3, 4, 7].

It is interesting to note that in the Schwarzschild case, the fluid is at rest ($\beta = 0$) at the spatial infinity and its velocity reaches the speed of light ($\beta = 1$) at the origin. The

opposite case is represented by the de Sitter model where $\beta = 0$ is at the origin and $\beta = 1$ at the spatial infinity. One can consider a more nontrivial geometry of Reissner-Nordström-de Sitter with $f = 1 - r_0/r + e^2/r^2 - \Lambda r^2/3$. Here the two (the black and the cosmological) horizons are present, and the corresponding hydrodynamical model describes a flow which reaches the velocity of light both at the origin and at the spatial infinity.

A generalization of the above results to a rotating black hole is nontrivial. The fluid in this case should move not only radially but with an additional axial twist.

It may be noticed that the coordinate transformation (12), (18), (20) is very much analogous to the other regular transformations on the horizon, see [8, 9, 10], for example. This point is emphasized in the recent paper [11] which essentially overlaps with our observations.

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